Technical Notes

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Rotational Equations of Motion for a Triaxial Rigid Body

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A SET of differential equations which govern the rotational motion of a uniaxial, rigid body under nonconservative torques has been obtained by Crenshaw and Fitzpatrick.[1] These equations, which were derived in a rather elaborate manner using canonical transformations and Poisson brackets, are suitable for numerical integration and offer advantages in speed and accuracy over the usual form of Euler's equations, when the magnitude of the total external torque is small compared to the magnitude of the rotational angular momentum of the body. Some artificial satellites and space vehicles fall into this category.

Equations (23) and (24) of Ref. 1 are termed perturbation equations by the authors, because the solution to the problem of a freely rotating, uniaxial, rigid body was used in deriving them from canonical perturbation equations. It turns out, however, that Eqs. (23) and (24) may be derived without using the free-motion solution and the associated canonical perturbation equations, or any other auxiliary device. The cited equations are therefore not perturbation equations, since no "unperturbed" solution[2] must be perturbed to obtain them. They are, more correctly, Euler's equations and Poisson's equations written in rather unconventional forms.

The main purpose of this Note is to derive six first-order differential equations which govern the rotational motion of a triaxial rigid body about its center of mass under arbitrary torques. The derivation of these equations starts from first principles and relies on no auxiliary device. The equations given here are similar in form to Eqs. (23) of Ref. 1, offer the same advantages, and reduce to those equations when the body is made axisymmetric.

This Note's secondary purpose is to give three alternate equations which do not exhibit a certain type of singularity present in three of the original triaxial equations.

Triaxial Equations

The rotational angular momentum $\mathbf{H}$ of a rigid body about its center of mass may be defined by its magnitude h and the two angles ψ^* and θ^*, as shown in Fig. 1, where the $0XYZ$ system is nonrotating and 0 is the center of mass of the body. The time rate of change of $\mathbf{H}$, as resolved in the $0xyz$ system (the angular momentum system), in matrix notation is

$$\dot{\mathbf{H}} = \delta\mathbf{H}/\delta t + \mathbf{\Omega}\mathbf{H} = \mathbf{T}, \tag{1}$$

where

$$\mathbf{H} = (0\ 0\ h)^T \tag{2a}$$

$$\delta\mathbf{H}/\delta t = (0\ 0\ \dot{h})^T \tag{2b}$$

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -\dot{\psi}^* c\theta^* & \dot{\psi}^* s\theta^* \\ \dot{\psi}^* c\theta^* & 0 & -\dot{\theta}^* \\ -\dot{\psi}^* s\theta^* & \dot{\theta}^* & 0 \end{bmatrix} \tag{2c}$$

$$\mathbf{T} = (T_x\ T_y\ T_z)^T \tag{2d}$$

and the superscript T denotes the transpose of a matrix so superscripted. In Eqs. (2), $c \equiv$ cosine and $s \equiv$ sine, respectively, and T_x, T_y, and T_z are components of the external torque $\mathbf{T}$ about 0. Equating terms in Eq. (1), we get

$$\dot{h} = T_z \tag{3a}$$

$$\dot{\theta}^* = -T_y/h \tag{3b}$$

$$\dot{\psi}^* = T_x/hs\theta^*,\ h \neq 0 \tag{3c}$$

Since the angular velocity $\boldsymbol{\omega}^H$ of the $0xyz$ system, resolved in that system, is

$$\boldsymbol{\omega}^H = (\dot{\theta}^*\ \dot{\psi}^* s\theta^*\ \dot{\psi}^* c\theta^*)^T \tag{4}$$

it follows, from Eqs. (3) and (4), that

$$\boldsymbol{\omega}^H = (-T_y\ T_x\ T_x\cot\theta^*)^T/h \tag{5}$$

Let $0x'y'z'$ denote a principal axis system for the rigid body at its center of mass, and let $I_{x'}$, $I_{y'}$, and $I_{z'}$ denote the associated principal moments of inertia. The $0x'y'z'$ system is oriented with respect to the $0xyz$ system using the Euler angles ϕ^*, θ', and ϕ' as shown in Fig. 2.

The total angular velocity $\boldsymbol{\omega}$ of the body about 0, as resolved in the $0x'y'z'$ system, is

$$\boldsymbol{\omega} = \boldsymbol{\omega}^R + \mathbf{B}\boldsymbol{\omega}^H \tag{6}$$

where

$$\boldsymbol{\omega}^R = \mathbf{A}\dot{\mathbf{\Theta}} \tag{7a}$$

$$\mathbf{A} = \begin{pmatrix} c\phi' & -s\phi' s\theta' & 0 \\ -s\phi' & c\phi' s\theta' & 0 \\ 0 & c\theta' & 1 \end{pmatrix} = \begin{pmatrix} c\phi' & s\phi' & 0 \\ -s\phi' & c\phi' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & s\theta' & 0 \\ 0 & c\theta' & 1 \end{pmatrix} \tag{7b}$$

$$\dot{\mathbf{\Theta}} = (\dot{\theta}'\ \dot{\phi}^*\ \dot{\phi}')^T \tag{7c}$$

$$\mathbf{B} = \begin{pmatrix} c\phi' & s\phi' & 0 \\ -s\phi' & c\phi' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c\theta' & s\theta' \\ 0 & -s\theta' & c\theta' \end{pmatrix}\begin{pmatrix} c\phi^* & s\phi^* & 0 \\ -s\phi^* & c\phi^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7d}$$

Also, the rotational angular momentum of the body about its center of mass may be expressed, in the $0x'y'z'$ system, in the

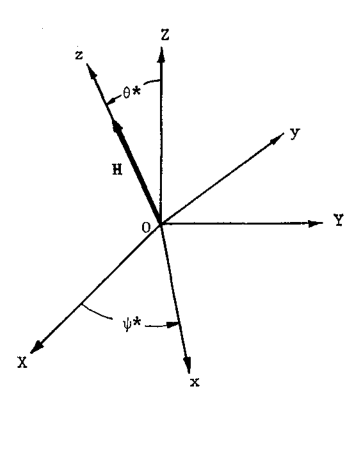

Fig. 1 Angular momentum system.

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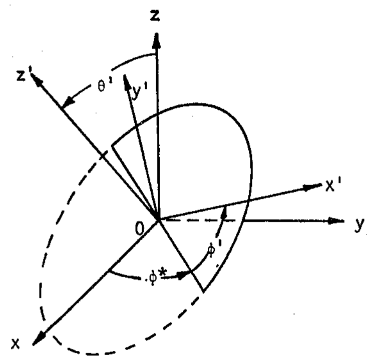

Fig. 2 Principal axes of the triaxial body, 3-1-3 rotation sequence.

two forms

$$\mathbf{H} = \mathbf{I}\boldsymbol{\omega} \tag{8}$$

and

$$\mathbf{H} = \mathbf{A}(0 \ \mathrm{h} \ 0)^T \tag{9}$$

where

$$\mathbf{I} = \begin{pmatrix} I_{x'} & 0 & 0 \\ 0 & I_{y'} & 0 \\ 0 & 0 & I_{z'} \end{pmatrix} \tag{10}$$

Equating Eqs. (8) and (9), using Eq. (6), and solving for $\dot{\boldsymbol{\Theta}}$, we obtain

$$\dot{\boldsymbol{\Theta}} = \mathbf{A}^{-1}\mathbf{I}^{-1}\mathbf{A}(0 \ h \ 0)^T - \mathbf{A}^{-1}\mathbf{B}(-T_y \ T_x \ T_x\cot\theta^*)^T/h \tag{11}$$

where

$$\mathbf{A}^{-1} = \csc\theta' \begin{pmatrix} s\theta' & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c\theta' & s\theta' \end{pmatrix} \begin{pmatrix} c\phi' & -s\phi' & 0 \\ s\phi' & c\phi' & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

and, if we let $J_{x'} = 1/I_{x'}$, $J_{y'} = 1/I_{y'}$, and $J_{z'} = 1/I_{z'}$,

$$\mathbf{I}^{-1} = \begin{pmatrix} J_{x'} & 0 & 0 \\ 0 & J_{y'} & 0 \\ 0 & 0 & J_{z'} \end{pmatrix}$$

Expansion of the matrices appearing in Eq. (11) leads to the equations

$$\dot{\theta}' = J_{x'}J_{y'}(I_{y'} - I_{x'})hs\theta's\phi'c\phi' - (T_{\phi'}\csc\theta' - T_{\phi^*}\cot\theta^*)/h \tag{13a}$$

$$\dot{\phi}^* = (J_{x'}s^2\phi' + J_{y'}c^2\phi')h - (T_{\theta'}\cot\theta' + T_{\theta^*}\cot\theta^*)/h \tag{13b}$$

$$\dot{\phi}' = (J_{z'} - J_{x'}s^2\phi' - J_{y'}c^2\phi')hc\theta' + T_{\theta'}\csc\theta'/h \tag{13c}$$

where in order to compare our results with those obtained by Crenshaw and Fitzpatrick, we have introduced the following notation:

$$T_{\phi'} \equiv T_{z'} = T_z c\theta' + (T_x s\phi^* - T_y c\phi^*)s\theta' \tag{14a}$$

$$T_{\theta'} \equiv T_x c\phi^* + T_y s\phi^* \tag{14b}$$

$$T_{\phi^*} \equiv T_z \tag{14c}$$

$$T_{\theta^*} \equiv T_x \tag{14d}$$

$$T_{\psi^*} \equiv T_Z = T_z c\theta^* + T_y s\theta^* \tag{14e}$$

Equations (13) along with

$$\dot{h} = T_{\phi^*} \tag{15a}$$

$$\dot{\theta}^* = (T_{\phi^*}\cot\theta^* - T_{\psi^*}\csc\theta^*)/h \tag{15b}$$

$$\dot{\psi}^* = T_{\theta^*}\csc\theta^*/h \tag{15c}$$

reduce to Crenshaw and Fitzpatrick's[1] Eqs. (23) when $I_{x'} = I_{y'}$. Equations corresponding to Eqs. (24) of Ref. 1 may also be obtained in a similar manner, if one refers the motion of the $0xyz$ system to a coordinate system which itself rotates with a certain prescribed angular velocity relative to the $0XYZ$ system.

Notice that since

$$(d/dt)(hc\theta') = T_{\phi^*}c\theta' - hs\theta'\dot{\theta}' \tag{16a}$$

$$(d/dt)(hc\theta^*) = T_{\phi^*}c\theta^* - hs\theta^*\dot{\theta}^* \tag{16b}$$

the equations

$$\dot{P}_{\phi'} = J_{x'}J_{y'}(I_{x'} - I_{y'})(P_{\phi^*}{}^2 - P_{\phi'}{}^2)s\phi'c\phi' + T_{\phi'} \tag{17a}$$

$$\dot{P}_{\phi^*} = T_{\phi^*} \tag{17b}$$

$$\dot{P}_{\psi^*} = T_{\psi^*} \tag{17c}$$

where we have defined

$$P_{\phi'} \equiv hc\theta', z'\text{—axis component of } \mathbf{H} \tag{18a}$$

$$P_{\phi^*} \equiv h, \text{ magnitude of } \mathbf{H} \tag{18b}$$

$$P_{\psi^*} \equiv hc\theta^*, Z\text{—axis component of } \mathbf{H} \tag{18c}$$

may be used in place of Eqs. (13a), (15a), and (15b), respectively.

It may be noted that if the torques acting on the body are conservative, Eqs. (13) and (17) form a canonical set.[3]

Alternate Equations

It is obvious that Eqs. (13) and (15) possess singularities when either θ' or θ^* is zero. This is a well-known difficulty associated with the use of Euler angles. The problem with θ^* can almost always be remedied by a judicious choice of the reference system $0XYZ$, while the problem involving θ' can be solved in some cases by using variations in the sequence of rotations from the $0xyz$ system to the $0x'y'z'$ system. As an example, we consider using a 3-2-1 rotation sequence as shown in Fig. 3, rather than the 3-1-3 rotation sequence shown in Fig. 2. Use of the 3-2-1 sequence alters the expressions for $\boldsymbol{\omega}^R$, $\mathbf{A}$, and $\mathbf{B}$ and leads to the equations

$$\dot{\phi}' = -(J_{x'} - J_{y'}s^2\phi' - J_{z'}c^2\phi')hs\theta' + T_{\theta'}\sec\theta'/h \tag{19a}$$

$$\dot{\phi}^* = (J_{y'}s^2\phi' + J_{z'}c^2\phi')h + T_{\theta'}\tan\theta'/h - T_{\theta^*}\cot\theta^*/h \tag{19b}$$

$$\dot{\theta}' = J_{y'}J_{z'}(I_{z'} - I_{y'})\mathrm{h}c\theta's\phi'c\phi' - T_{\theta'}\sec\theta'/h - T_{\theta^*}\tan\theta'/h \tag{19c}$$

which are not singular when $\theta' = 0$.

The last two of Eqs. (17) are also valid in this case, while the first of Eqs. (17) should be replaced by

$$\dot{P}_{\phi'} = J_{y'}J_{z'}(I_{y'} - I_{z'})(P_{\phi^*}{}^2 - P_{\phi'}{}^2)s\phi'c\phi' + T_{\phi'} \tag{20}$$

where we have redefined $P_{\phi'}$ as follows:

$$P_{\phi'} \equiv -hs\theta' \tag{21}$$

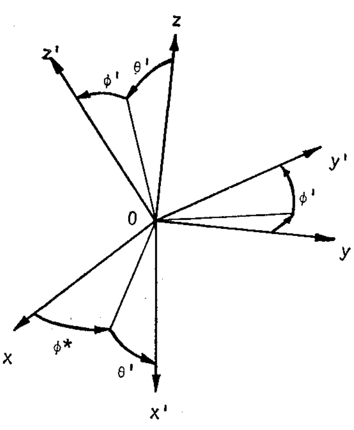

Fig. 3 Principal axes of the triaxial body, 3-2-1 rotation sequence.

Summary

First-order differential equations which govern the rotational motion of a triaxial rigid body under the influence of arbitrary external torques have been derived for two different Euler rotation sequences. The derivations are very straightforward, relying in no way on having a solution to the free-motion problem. The equations given here are suitable for numerical integration and offer advantages in speed and accuracy over the usual Euler's equations when the magnitude of the total external torque is small compared to the magnitude h of the rotational angular momentum $\mathbf{H}$; for they take full advantage of the fact that in such a case $\mathbf{H}$ is an almost constant vector. Since the solution to the problem of describing the rotational motion of a free, triaxial, rigid body with respect to a set of axes one of which lies along $\mathbf{H}$ is well-known,[4] the equations derived in this Note are also suitable for use in studying the effects of perturbing torques on the rotational motion of artificial satellites.[1,5]

References

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Shock-Wave Strength for Separation of a Laminar Boundary Layer at Transonic Speeds

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Stewartson and Williams[1] have derived asymptotic representations which describe the shock-induced separation of a supersonic laminar boundary layer. Their analysis expresses in a systematic way the idea, proposed in the same context by Lighthill,[2] that an abrupt change in pressure causes changes in the boundary-layer profile which are described approximately by inviscid-flow equations, except in a very thin viscous sublayer where the boundary-layer equations are retained in the first approximation. The results of Ref. 1 are not uniformly valid when the external-flow Mach number M_∞ is close to one, because linear supersonic theory has been used to relate the pressure and the flow deflection just outside the boundary layer. According to transonic small-disturbance theory, the changes in pressure are of the same order as the two-thirds power of the flow deflection angle. Separation at transonic speeds will occur if this pressure rise is sufficient to overcome the viscous stresses acting on a fluid element very close to the surface and to cause a change in its velocity which is of the same order as its velocity just upstream of the pressure rise. Hence, as in Ref. 1, near the surface (i.e., in a very thin sublayer) the momentum equation should express a balance among viscous, pressure, and inertia forces, and the term representing acceleration should be nonlinear. The preceding ideas form the basis for the analysis outlined below, which follows very closely the derivation described in detail for a related problem in Ref. 5.

Let $\bar{x}/L$ and $\bar{y}/L$ be coordinates along and normal to the surface, respectively, made nondimensional with a characteristic length L (e.g., chord length). Stretched coordinates are defined as follows:

$$x = R^\alpha \bar{x}/L, \quad y = R^{1/2}\bar{y}/L, \quad Y = R^\beta \bar{y}/L = R^{\beta-1/2}y \quad (1)$$

where α and β are to be determined and R is the Reynolds number $\bar{\rho}_w \bar{u}_\infty L/\bar{\mu}_w$ based on the freestream velocity and on thermodynamic properties at the surface. Throughout most of the boundary layer the flow will be described by small perturbations on an initial velocity profile $\bar{u}/\bar{u}_\infty = u_0(y)$. Since $u_0(y) = O(y)$ as $y \to 0$, the sublayer velocity will be $O(R^{1/2-\beta}Y)$ as $Y \to \infty$. Just outside the boundary layer (for $y \to \infty$ but $\bar{y}/L \to 0$) $p = O(\theta^{2/3})$, where p is the relative change in pressure from the value just upstream of the disturbance and θ is the flow deflection angle. It will be anticipated that $p_y \sim 0$ throughout the boundary layer, and the terms $\partial(\bar{\rho}\bar{u})/\partial\bar{x}$ and $\partial(\bar{\rho}\bar{v})/\partial\bar{y}$ appearing in the continuity equation will be required to be of the same order. These considerations suggest the following assumed asymptotic expansions:

$$\bar{u}/\bar{u}_\infty \sim u_0(y) + \lambda_1(R)u_1(x,y) + \ldots \quad (2a)$$

$$\bar{v}/\bar{u}_\infty \sim R^{\alpha-1/2}\lambda_1(R)v_1(x,y) + \ldots \text{ (main b.l.)} \quad (2b)$$

$$\bar{u}/\bar{u}_\infty \sim R^{-\beta+1/2}U_1(x,Y) + \ldots \quad (3a)$$

$$\bar{v}/\bar{u}_\infty \sim R^{\alpha-2\beta+1/2}V_1(x,Y) + \ldots \text{ (sublayer)} \quad (3b)$$

$$\gamma^{-1}p \sim \epsilon p_1(x) + \ldots \quad (4)$$

where γ = ratio of specific heats; $\lambda_1(R) \ll 1$; and $R^{\alpha-1/2}\lambda_1 = O(\epsilon^{3/2})$ for the transonic case. The first term in the density is $\bar{\rho}/\bar{\rho}_w \sim \rho_0(y)$, and in the sublayer $\bar{\rho}/\bar{\rho}_w \sim \rho_0(0) = 1$. Substitution in the x-momentum equation then gives, for the main part of the boundary layer and the sublayer, respectively,

$$\lambda_1 R^\alpha \rho_0(u_0 u_{1_x} + v_1 u_0') = -\epsilon R^\alpha p_1' + \ldots \quad (5)$$

$$R^{-2\beta+1+\alpha}(U_1 U_{1_x} + V_1 U_{1Y}) = -\epsilon R^\alpha p_1' + R^{\beta-1/2}U_{1YY} + \ldots \quad (6)$$

In the analogous derivation of Ref. 1, $R^{\alpha-1/2}\lambda_1 = O(\epsilon)$ and a formulation for $\alpha < \frac{1}{2}$ is sought because it is anticipated that the pressure change takes place over a distance large compared with $R^{-1/2}$. Therefore the pressure gradient is absent from the first approximation to Eq. (5), and the solution for v_1 has the form $v_1(x,y) = u_0(y)G_1'(x)$, implying that the flow deflection $\theta \sim R^{\alpha-1/2}\lambda_1 v_1/u_0$ is independent of y and is a consequence only of the displacement effect of the sublayer. Requiring that the flow deflection remain of the same order in the sublayer, so that $\lambda_1 R^{\alpha-1/2} = O(R^{\alpha-\beta})$, and equating the orders of the three terms in Eq. 6, one obtains the results given in Ref. 1: $\alpha = 3/8$, $\beta = 5/8$, $\epsilon = O(R^{-1/4})$, $\lambda_1 = O(R^{-1/8})$; the results for ϵ and for α were proposed originally by Lees[3] and by Lighthill,[2] respectively.

At transonic speeds, with $\lambda_1 R^{\alpha-1/2} = O(\epsilon^{3/2})$, the corresponding results are $\alpha = \frac{3}{10}$, $\beta = \frac{3}{5}$, $\epsilon = O(R^{-1/5})$, $\lambda_1 =$

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